Elastic settlement of arbitrarily shaped foundations embedded in half-space

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The settlement of foundations under working loads which induce relatively small strains in the soil is frequently estimated in practice on the basis of results from the theory of elasticity. Despite its shortcomings in modelling actual soil behaviour, elastic theory may be quite useful especially in predicting immediate settlements on saturated clayey deposits, provided that an appropriate value of undrained secant Young's modulus has been experimentally determined.

Most of the available elastic solutions refer to loads acting directly on the ground surface. To estimate the settlement of an embedded foundation without resorting to expensive numerical (mainly finite element and boundary element) techniques, engineers apply reduction factors to the settlement of the corresponding surface foundation. While most available reduction factors are purely empirical in nature, an approximate solution by Fox (1948) has been particularly popular for embedded flexible rectangular foundations. Fox's solution has been reproduced in the form of a simple chart by Janbu, Bjerrum & Kjaernsli (1956). This chart is still widely used in geotechnical engineering, being recommended in several textbooks of soil mechanics and foundation engineering. Unfortunately, as Burland (1970) and Christian & Carrier (1978) have shown, Fox's factors may grossly exaggerate the effect of embedment, when compared with the results of finite element analyses. This is hardly surprising: Fox's results, based on Mindlin's solution for a point load within a half-space, refer to a uniformly loaded area *surrounded* by (and bonded to) an elastic continuum, rather than to an embedded foundation. That is, Fox's results implicitly assume that part of the carried load is transmitted to the ground through tension between the upper side of the foundation mat and the overlying soil. However, no (net) tension can usually develop between soil and foundation and, in most cases, the 'overlying' soil has been removed by excavation.

The results of the studies by Burland and Christian and Carrier are limited to uniformly loaded areas placed at the bottom of an open trench. Rigid foundations were not considered and the effects of contact between the soil and the foundation sidewalls were neglected. However, Kaldjian (1969), Johnson, Christiano & Epstein (1975) and Kausel & Ushijima (1979) have presented some results for circular and strip foundations with vertical sides in perfect contact with the soil.

In this Note a simple realistic analytical expression is developed for estimating the vertical elastic settlement of arbitrarily shaped rigid foundations embedded in a reasonably uniform and deep soil deposit, modelled as a homogeneous half-space (Fig. 1). The expression is applicable to a large range of embedment depths and a variety of solid base shapes, ranging from circular to strip and including rectangles of any aspect ratio as well as odd shapes differing substantially from rectangular or circular. (Annular base shapes are excluded, however.) Further, the expression encompasses the wide variation in the type of contact between foundation sidewalls and surrounding soil, between the extreme cases of complete perfect contact and of no contact at all.

The development of the proposed expression was based on an improved qualitative understanding of the effects of embedment, substantiated quantitatively by the results of extensive rigorous parametric studies using the boundary element method, and including numerous analytical and numerical results available in the literature. It is emphasized that the proposed algebraic expressions are curve fits of the analytical data; hence, the accuracy of the settlement computed from the proposed expression may be not better than 10–20%. Note, however, that discrepancies of about 10% or more have also been observed among the results obtained from different rigorous solutions to the same problem. Such discrepancies arise because of the different assumptions regarding the behaviour of the soil–foundation interface ('rough' versus 'relaxed' boundary conditions), different solution methods (such as integral transform techniques, semi-analytical procedures, finite elements and the boundary element method) and different de-

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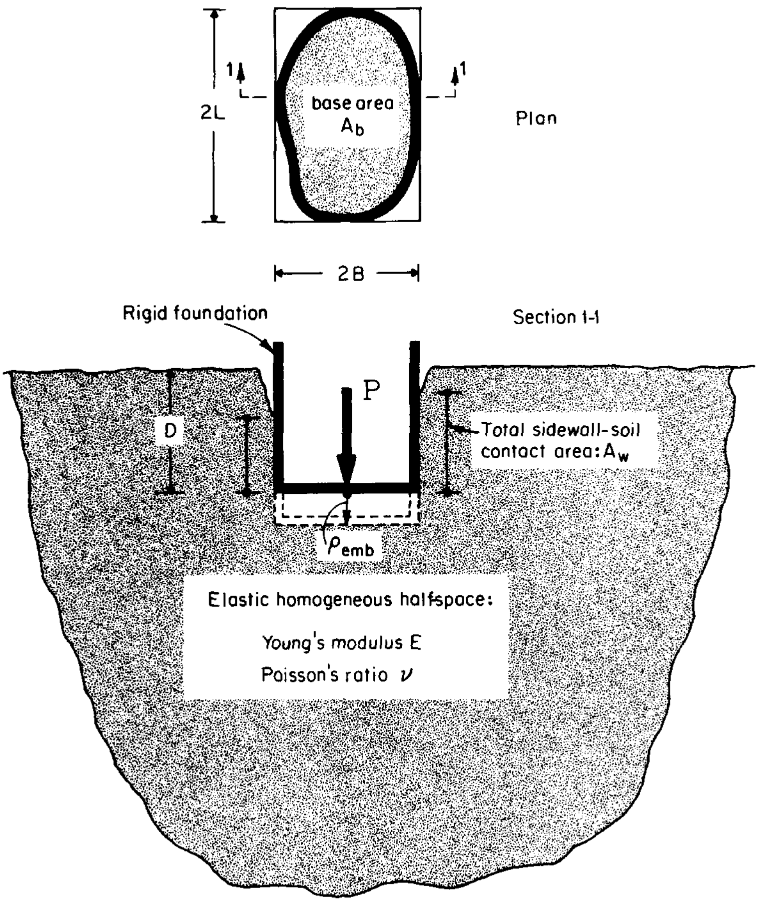


Fig. 1. Problem geometry

grees of precision with which the calculations have been carried out.

SETTLEMENT OF SURFACE FOUNDATIONS

The elastic settlement of a rigid foundation of arbitrary shape resting on the surface of a homogeneous half-space of modulus E and Poisson's ratio ν, and carrying a total vertical force P may be estimated from

$$\rho_{sur} = \frac{P}{EL}(1-\nu^2)\mu_{shape} \tag{1a}$$

in which

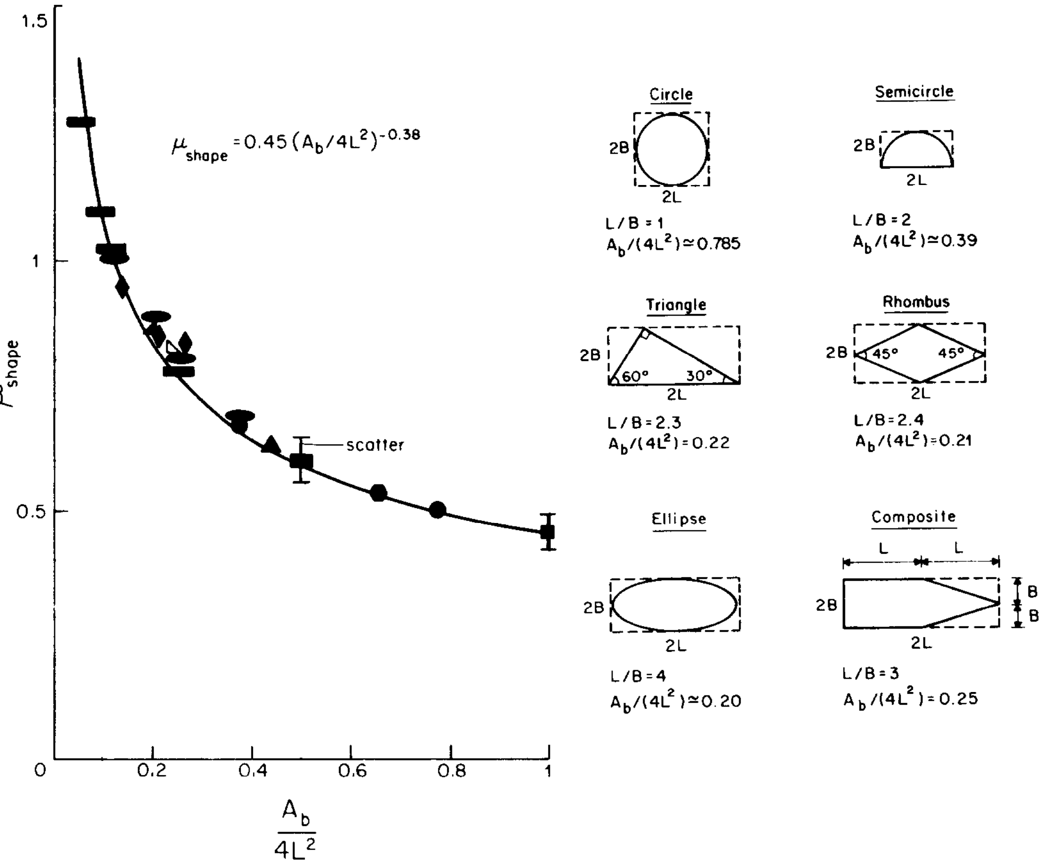
$$\mu_{shape} = 0{\cdot}45\left(\frac{A_b}{4L^2}\right)^{-0{\cdot}38} \tag{1b}$$

where A_b is the plan area of the foundation–soil contact surface, and $2L$ and $2B$ are the length and width of the rectangle circumscribed to the actual contact surface, as illustrated in Figs 1 and 2(b). The dimensionless shape parameter $A_b/4L^2$ is equal to unity for a square, 0·785 for a circle and zero for an infinitely long strip. For rectangles $A_b/4L^2 = B/L$, i.e. it reduces to the inverse of the aspect ratio.

Expression (1b) has been derived by fitting analytical and numerical results from the numerous publications listed in Table 1, including several results by the Authors. As shown in Fig. 2(a), the scatter of the results around the curve of equation (1b) is very small for all the cases considered. For an arbitrary solid shape (whose actual area A_b is not less than about 40% of the area $4BL$ of the circumscribed rectangle) the true elastic surface settlement is expected to be within 5–10% of the value predicted by equation (1).

EFFECTS OF EMBEDMENT ON FOUNDATION SETTLEMENT

There are three possible effects of embedment on the vertical settlement of a rigid foundation.



(a) (b)

Fig. 2. (a) Surface settlement shape factor versus foundation shape parameter (the shapes of the data points correspond approximately to those of Table 1) and (b) examples of non-rectangular base shapes and corresponding circumscribed rectangles

First of all, for the usual situation in which soil stiffness increases with depth, embedded foundations transmit load to stiffer soil beneath them than do surface foundations. Hence, other things being equal, smaller settlement is expected for an embedded foundation, although settlements may be the same or even increase compared with surface foundations if the soil stiffness decreases with depth. This important effect of embedment will not be further addressed herein. However, when applying the proposed method to a practical situation

(*a*) it must be ensured that the soil deposit is indeed reasonably homogeneous and deep
(*b*) a representative value of soil modulus for the particular depth of embedment must be established.

The other two effects which modify the behaviour of embedded foundations and are addressed in this Note are referred to as the 'trench' effect and the 'sidewall contact' effect. Both of these effects are illustrated schematically in Fig. 3.

The trench effect refers to the fact that, even in a perfectly homogeneous half-space, the settlement of a foundation placed at the bottom of an open trench is smaller than that of the same foundation placed on the ground surface. To understand why, visualize a horizontal plane surface passing through the foundation base. For the surface foundation this plane deforms free of any external stress. However, for the embedded foundation, normal and shear stresses from the overlying soil restrict the movement of that horizontal plane (Fig. 3(a)), thereby reducing the foundation settlement from ρ_{sur} to $\rho_{trench} = \mu_{trench}\rho_{sur}$. There is analytical as well as experi-

Table 1. Surface foundations: data of Fig. 2(a) and their sources*

Shape	L/B	$A_b/4L^2$
Square	1	1
Circle	1	0·79
Hexagon (regular)	1	0·65
Rectangle	2	0·5
Triangle (equilateral)	1·15	0·44
Ellipse	2	0·39
Semicircle	2	0·39
Ellipse	3	0·26
Triangle (45°–45°–90°)	2	0·25
Rhombus (60°)	2	0·25
Rectangle	4	0·25
Triangle (30°–60°–90°)	2·3	0·25
Rhombus (45°)	2·4	0·22
Ellipse	4	0·21
Rhombus (30°)	3·7	0·20
Ellipse	6	0·14
Rectangle	8	0·13
Rectangle	10	0·10
Rectangle	20	0·05
Strip	∞	0

* References: Gorbunov-Possadov & Serebrajanyi (1961); Dominguez & Roesset (1978); Selvadurai (1979); Barkan (1962); Borodachev (1964); Sneddon (1951); Conway & Farnham (1968); Savidis (1977); Rücker (1982); Muskhelishvili (1953); Poulos & Davis (1974); Authors.

mental evidence supporting the existence of the trench effect. Erden's (1974) experimental work on dry sand revealed that 'even with no side contact along the embedment depth, the effect of embedment [to reduce the settlement] is not caused solely by an increase in shear modulus beneath the footing'. Moreover, Erden (1974) showed that the settlement of a foundation in an open trench is essentially the same as, and in fact even somewhat smaller than, that of a surface foundation when the free ground surface is pneumatically loaded with a normal pressure equivalent to the overburden stress at basemat level. Apparently, both normal *and* shear 'overburden' stresses, as sketched in Fig. 3(a), play a role in reducing the trench settlement. The numerical data needed to develop a relationship for $\mu_{\text{trench}} = \rho_{\text{trench}}/\rho_{\text{sur}}$ have been compiled from several publications (listed in Table 2), dealing primarily with circular and strip foundations, and from a comprehensive parametric study by the Authors for embedded circular, strip and rectangular foundations with aspect ratios of up to 6, and for two values of Poisson's ratio: 0·33 (typical of cohesionless soil) and 0·49 (typical of saturated clays). Fig. 4 presents this information in the form of a graph of μ_{trench} as a function of the dimensionless parameter $(D/B)[1+\frac{4}{3}\times(A_b/4L^2)]$. Determined by trial and error so that the results plot within a relatively narrow band, this parameter combines the relative depth of embedment and the base shape. An approximate expression for the trench effect coefficient μ_{trench} was developed by passing a straight line through the analytical data points

$$\mu_{\text{trench}} = 1 - 0{\cdot}04\frac{D}{B}\left(1+\frac{4}{3}\frac{A_b}{4L^2}\right) \tag{2}$$

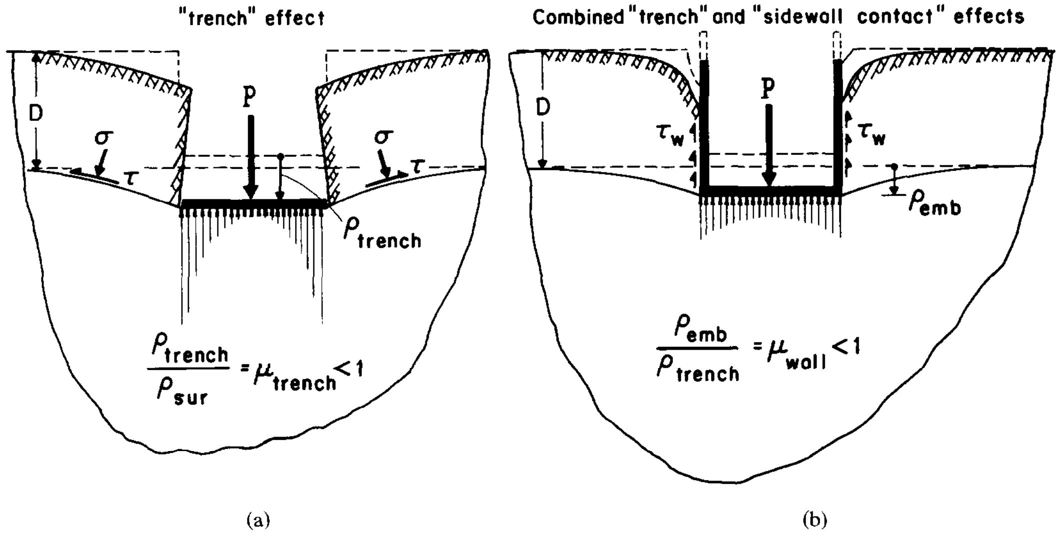


Fig. 3. Embedded foundation: schematic illustration of trench and sidewall contact effects

Table 2. Embedded foundations: data of Figs 4 and 5, and their sources

Symbol	ν	Shape*	Reference
⊙	0·25	Circle	Kaldjian (1969)
◍ ⊘	0·25 0·49	Circle (uniform applied pressure)	Burland (1970), Christian & Carrier (1978)
■	0·33	Square	Dominguez & Roesset (1978)
●	0·33	Circle (partial sidewall contact)	Tassoulas (1981)
◒	0·40	Circle	Authors
◪	0·33	Rectangle $L/B = 4$	Authors
▭ ▬	0·33 0·49	Rectangle $L/B = 6$	Authors
⊡ ▢ ⬬	0·25 0·33 0·49	Strip	Johnson *et al.* (1975); Authors

* Unless otherwise stated, results are for a rigid base and a rigid sidewall.

The error of equation (2) is in most cases less than 10%.

The third phenomenon to be considered is the sidewall contact effect. When the vertical sidewalls of an embedded foundation are in contact with the surrounding soil, part of the applied load is transmitted to the ground through shear tractions acting along the vertical sides of the wall (Fig. 3(b)). This additional load path leads to a further reduction in the settlement of an embedded foundation with sidewall contact, compared with that in an open trench.

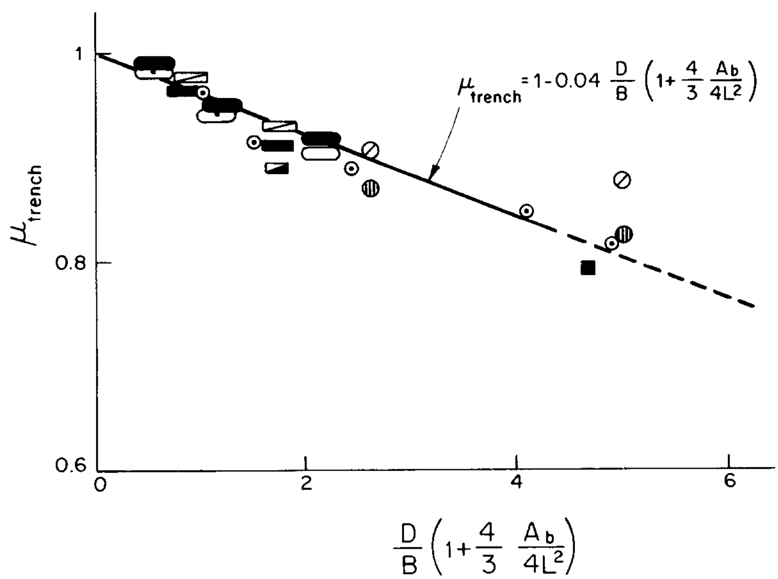


Fig. 4. Trench settlement reduction factor versus shape-dependent normalized depth parameter (for references see Table 2)

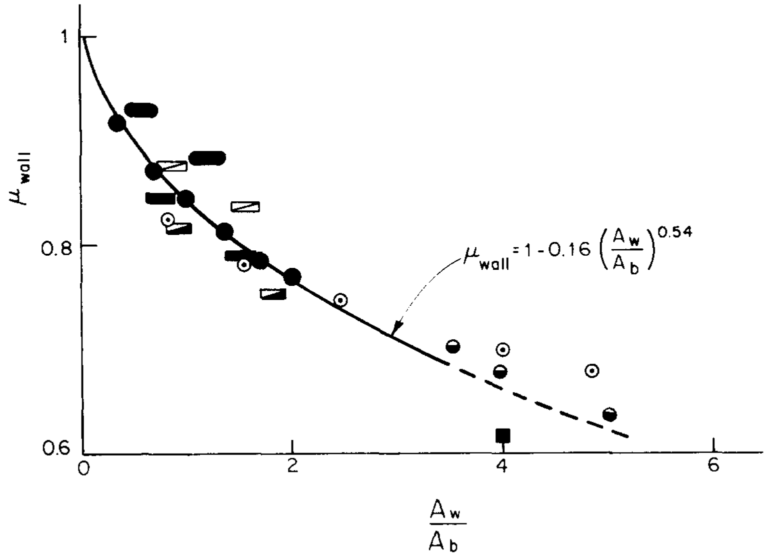


Fig. 5. Sidewall contact settlement reduction factor versus wall–base area ratio (for references see Table 2)

This reduced settlement ρ_{emb} is evaluated from the open trench settlement ρ_{trench} through the sidewall coefficient μ_{wall}

$$\rho_{emb} = \rho_{trench}\mu_{wall} \tag{3}$$

The numerical data for developing an expression for μ_{wall} were obtained by dividing the respective settlements of an embedded foundation with complete sidewall–soil contact (ρ_{emb}) and with absolutely no contact (ρ_{trench}). The sources of these data are also included in Table 2.

Figure 5 shows the sidewall coefficient μ_{wall} as a function of the effective wall–base area ratio A_w/A_b, where A_w is the area of the sidewall–soil interface and A_b is the area of the basemat. The choice of the parameter A_w/A_b is quite natural, since effectively the coefficient μ_{wall} weighs the contributions to stiffness from the sides and from the base. Indeed, the scatter of the data points in Fig. 5 about the mean curve

$$\mu_{wall} = 1 - 0{\cdot}16\left(\frac{A_w}{A_b}\right)^{0{\cdot}54} \tag{4}$$

is fairly small and essentially independent of the basemat shape and of the relative depth.

In conclusion, the settlement of an embedded foundation may be approximated by

$$\rho_{emb} = \rho_{sur}\mu_{trench}\mu_{wall} \tag{5}$$

Substituting equations (1), (2) and (4)

$$\rho_{emb} = 0{\cdot}45\frac{P}{EL}(1-\nu^2)\left(\frac{A_b}{4L^2}\right)^{-0{\cdot}38} \times\left[1-0{\cdot}04\frac{D}{B}\left(1+\frac{4}{3}\frac{A_b}{4L^2}\right)\right] \times\left[1-0{\cdot}16\left(\frac{A_w}{A_b}\right)^{0{\cdot}54}\right] \tag{6}$$

is obtained for the settlement of a foundation embedded in a homogeneous half-space and having an arbitrary but solid base shape. For cases where there is doubt about the quality of contact between the sidewall and the surrounding soil, the engineer may want to apply a reduction factor q (between zero and unity) to the wall–base area ratio, i.e. to substitute qA_w/A_b for A_w/A_b in equation (6).

ILLUSTRATIVE EXAMPLE

Equation (6) is now directly applied to estimate the settlement of the embedded foundation portrayed in Fig. 6. In plan, the continuous and rigid foundation basemat is composed of two rectangles, one square and one semicircle. Part of the basemat perimeter is connected to a vertical sidewall 7·5 m high which is in contact with the surrounding soil. The sidewall–soil contact quality factor q is assumed to be 0·75. No sidewall (and hence no contact) exists along defgh, because of a hypothetical open space next to the foundation structure. The geometric parameters of the foundation and the elastic soil parameters are shown in Fig. 6. Of interest is

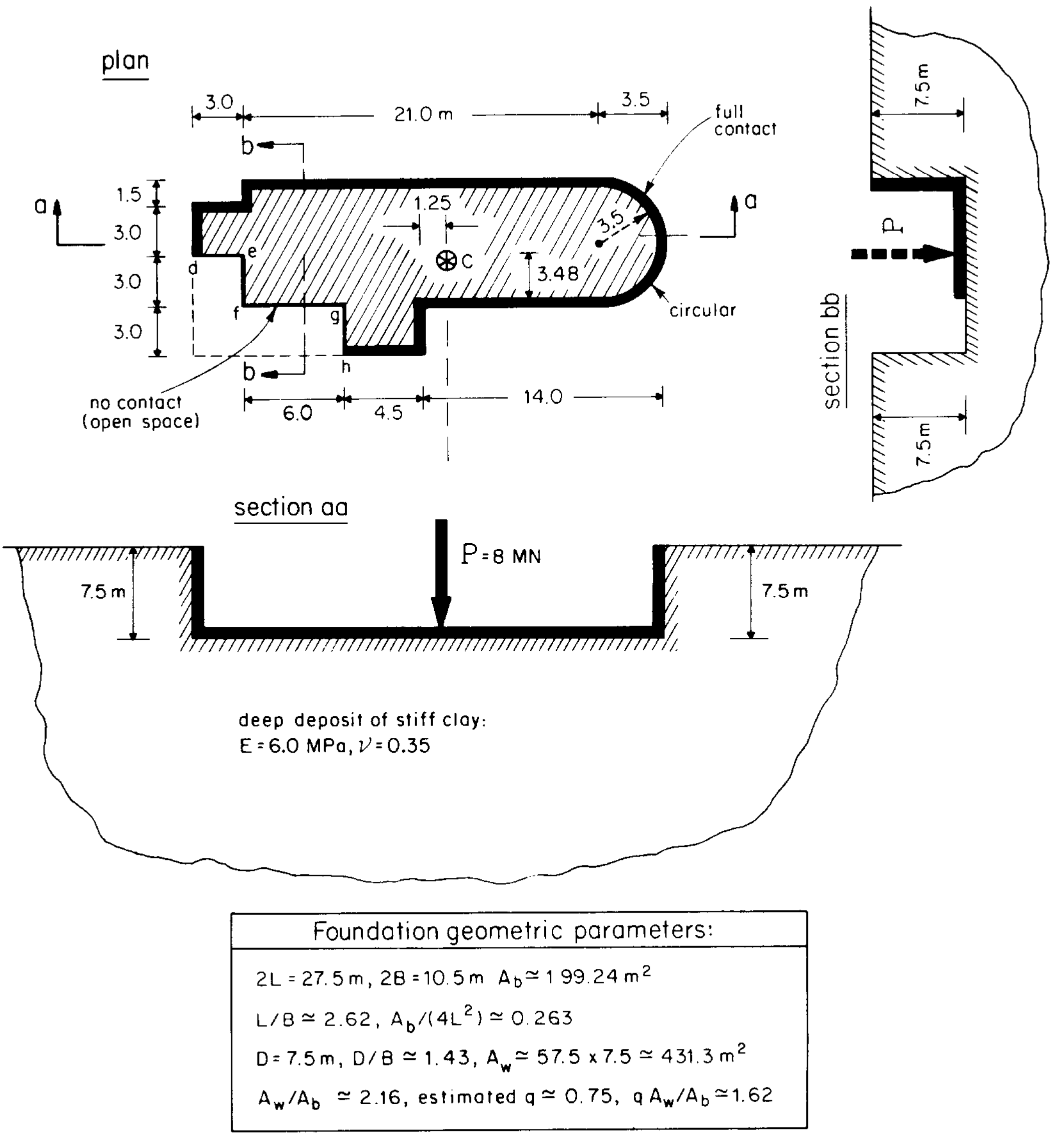


Fig. 6. Geometry and material parameters of an illustrative example

the settlement due to a central vertical force $P = 8$ MN. Ignoring as secondary the incidental rotation that may occur from the lack of complete symmetry of the soil reactions at the base and the walls, with respect to the centroid C of the base, equation (6) gives

$$\rho = \frac{8(1-0{\cdot}35^2)}{6\times 13{\cdot}75}\,0{\cdot}45\times 0{\cdot}263^{-0{\cdot}38}$$

$$\times\left[1-0{\cdot}04\times 1{\cdot}43\left(1+\frac{4}{3}\,0{\cdot}263\right)\right]$$

$$\times(1-0{\cdot}16\times 1{\cdot}62^{0{\cdot}54})$$

$$\approx 0{\cdot}085\times 0{\cdot}748\times 0{\cdot}923\times 0{\cdot}792$$

$$\approx 0{\cdot}046 \text{ m} = 46 \text{ mm}$$

CONCLUSION

A versatile simple analytical expression (equation (6)) is proposed for estimating the vertical elastic settlement of foundations with any solid basemat shape and embedded in reasonably deep and uniform soil deposit. The expression is applicable for a constant depth of embedment but encompasses all possible types of contact between sidewalls and surrounding soil: complete and perfect contact, partial contact and no contact at all.

It may be noted that, although the rigorous numerical and analytical results on which equation (6) is based were obtained for a homogeneous half-space, the same conceptual framework can be readily extended to the more realistic case of a soil profile consisting of a half-space below the base and of an overlying side soil (backfill) of a different material. Also, although most of the basic data relate to a perfectly rigid basemat, equation (6) may provide sufficiently accurate estimates of the average settlement of flexible foundations.

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